**The Gauss elimination game**

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This is actually a lecture on solving systems of equations using the method of *row reduction*, but it's more fun to formulate it in terms of a game.

To be specific, let's focus on a $2\times 2$ system (by "$2\times 2$" I mean 2 equations in the 2 unknowns $x, y$):

$$\left\{ \begin{array}{lcl} ax+by & = & r_1 \\ cx+dy & = & r_2 \end{array} \right. \tag{1}$$

Here $a, b, c, d, r_1, r_2$ are given constants. Putting these two equations down together means to solve them simultaneously, not individually. In geometric terms, you may think of each equation above as a line the the plane. To solve them simultaneously, you are to find the point of intersection (if it exists) of these two lines. Since $a, b, c, d, r_1, r_2$ have not been specified, it is conceivable that there are

- no solutions (the lines are parallel but distinct),
- infinitely many solutions (the lines are the same),
- exactly one solution (the lines are distinct and not parallel).

"Usually" there is exactly one solution. Of course, you can solve this by simply manipulating equations since it is such a low-dimensional system but the object of this lecture is to show you a method of solution which is "scalable" to "industrial-sized" problems (say $1000\times 1000$ or larger).

Strategy:

*Step 1*: Write down the *augmented matrix* of (1):

$$A = \begin{pmatrix} a & b & r_1 \\ c & d & r_2 \end{pmatrix}$$

This is simply a matter of stripping off the unknowns and recording the coefficients in an array. Of course, the system must be written in "standard form" (all the terms with "$x$" get aligned together, ...) to do this correctly.

*Step 2*: Play the Gauss elimination game (described below) to computing the row reduced echelon form of $A$, call it $B$ say.

*Step 3*: Read off the solution from the right-most column of $B$.

**The Gauss Elimination Game**

*Legal moves*: These actually apply to any $m \times n$ matrix $A$ with $m < n$.

1. $R_i \leftrightarrow R_j$: You can swap row $i$ with row $j$.

2. $cR_i \to R_i$ $(c \neq 0)$: You can replace row $i$ with row $i$ multiplied by any non-zero constant $c$. (Don't confuse this $c$ with the $c$ in (1)).

3. $cR_i + R_j \to R_i$ $(c \neq 0)$: You can replace row $i$ with row $i$ multiplied by any non-zero constant $c$ plus row $j$, $j \neq i$.

Note that move 1 simply corresponds to reordering the system of equations (1)). Likewise, move 2 simply corresponds to scaling equation $i$ in (1)). In general, these "legal moves" correspond to algebraic operations you would perform on (1)) to solve it. However, there are fewer symbols to push around when the augmented matrix is used.

*Goal*: You *win* the game when you can achieve the following situation. Your goal is to find a sequence of legal moves leading to a matrix $B$ satisfying the following criteria:

1. all rows of $B$ have leaading non-zero term equal to 1 (the position where this leading term in $B$ occurs is called a *pivot position*),

2. $B$ contains as many 0's as possible

3. all entries above and below a pivot position must be 0,

4. all entries below the diagonal of $B$ are 0.

This matrix $B$ is unique (this is a theorem which you can find in any text on elementary matrix theory or linear algebra) and is called the *row reduced echelon form* of $A$, sometimes written $rref(A)$.

Two comments: (1) If you are your friend both start out playing this game, it is likely your choice of legal moves will differ. That is to be expected. However, you must get the same result in the end. (2) Often if someone is to get "stuck" it is becuase they forget that one of the goals is to "kill as many terms as possible (i.e., you need $B$ to have as many 0's as possible). If you forget this you might create non-zero terms in the matrix while killing others. You should try to think of each move as being made in order to to kill a term. The exception is at the very end where you can't kill any more terms but you want to do row swaps to put it in diagonal form.

Now it's time for an example.

**Example** Solve

$$\left\{ \begin{array}{rcl} x + 2y & = & 3 \\ 4x + 5y & = & 6 \end{array} \right. \tag{2}$$

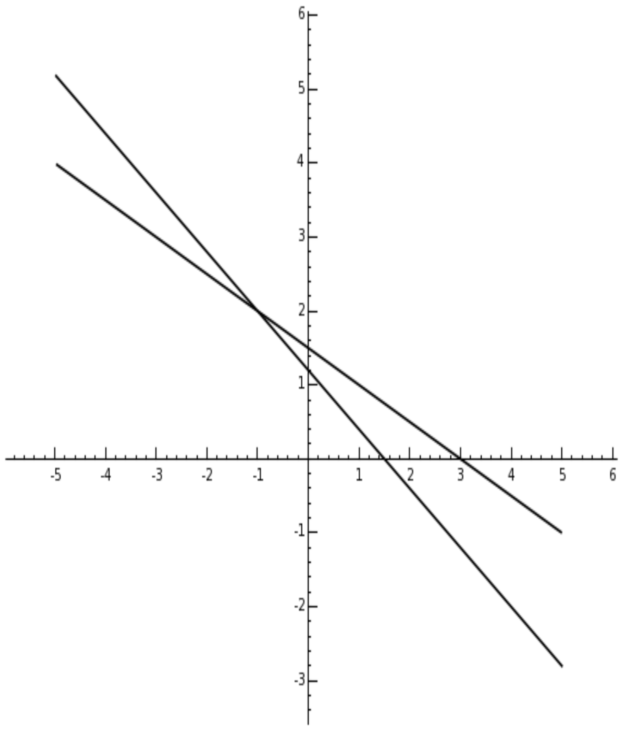


Figure 1: lines $x + 2y = 3$, $4x + 5y = 6$ in the plane.

The augmented matrix is

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

One sequence of legal moves is the following:

$-4R_1 + R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$

$-(1/3)R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

$-2R_2 + R_1 \to R_1$, which leads to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

Now we are done (we won!) since this matrix satisfies all the goals for a eow reduced echelon form.

The latter matrix corresponds to the system of equations

$$\begin{cases} x + 0y &= -1 \\ 0x + y &= 2 \end{cases} \tag{3}$$

Since the "legal moves" were simply matrix analogs of algebraic manipulations you'd appy to the system (2), the solution to (2) is the same as the solution to (3), whihc is obviously $x = -1, y = 2$. You can visually check this from the graph given above.

**Solving systems using inverses**

There is another method of solving "square" systems of linear equations which we discuss next.

One can rewrite the system (1) as a single matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

or more compactly as

$$A\vec{X} = \vec{r}, \tag{4}$$

where $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$. How do you solve (4)? The obvious this to do ("divide by $A$") is the right idea:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{X} = A^{-1}\vec{r}.$$

Here $A^{-1}$ is a matrix with the property that $A^{-1}A = I$, the identity matrix (which satisfies $I\vec{X} = \vec{X}$).

If $A^{-1}$ exists (and it usually does), how do we compute it? There are a few ways. One, if using a formula. In the $2 \times 2$ case, the inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

There is a similar formula for larger sized matrices but it is so unwieldy that is is usually not used to compute the inverse. In the $2 \times 2$ case, it is easy to use and we see for example,

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} = \frac{1}{-3} \begin{pmatrix} 5 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}.$$

A better way to compute $A^{-1}$ is the following. Compute the row reduced echelon form of the matrix $(A, I)$, where $I$ is the identity matrix of the same size as $A$. This new matrix will be (if the inverse exists) $(I, A^{-1})$. You can read off the inverse matrix from this.

Here is an example.

**Example** Solve

$$\begin{cases} x + 2y &= 3 \\ 4x + 5y &= 6 \end{cases}$$

using matrix inverses.

This is

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

To compute the inverse matrix, apply the Gauss elimination game to

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{pmatrix}$$

Using the same sequence of legal moves as in the previous example, we get

$-4R_1 + R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -4 & 1 \end{pmatrix}$

$-(1/3)R_2 \to R_2$, which leads to $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4/3 & -1/3 \end{pmatrix}$

$-2R_2 + R_1 \to R_1$, which leads to $\begin{pmatrix} 1 & 0 & -5/3 & 2/3 \\ 0 & 1 & 4/3 & -1/3 \end{pmatrix}$.

Therefore the inverse is

$$A^{-1} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}.$$

Now, to solve the system, compute

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$